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# A Noncommutative Approach to Ordinary Differential Equations

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We adapt ideas coming from Quantum Mechanics to develop a non-commutative strategy for the analysis of some systems of ordinary differential equations. We show that the solution of such a system can be described by an unbounded, self-adjoint and densely defined operator $H$ which we call, in analogy with Quantum Mechanics, the *Hamiltonian* of the system.

We discuss the role of $H$ in the analysis of the integrals of motion of the system. Finally, we apply this approach to several examples.

**KEY WORDS:** ordinary differential equations; quantum evolution.

## 1. INTRODUCTION

The mathematical analysis of systems of ordinary differential equations (SODE) is quite an old subject in pure and applied mathematics and there are not many aspects which still seem to require a deeper analysis: existence of solutions and their explicit analytic expressions, uniqueness, symmetries, are all been considered in hundreds of papers and books. However, in none of these papers our point of view has ever been considered, as far as we know. In this paper we will show that the solution of a SODE can be considered as a sort of quantum evolution of a fictitious closed quantum system and, for this reason, can be analyzed using the well established strategies discussed in all the textbooks of quantum mechanics (QM), (Cohen-Tannoudji *et al.*, 1977 Merzbacher, 1970; Schiff, 1968). Among the other results, this time evolution is implemented by unitary operators, and this feature simplifies many aspects of the usual approach to SODE.

Other advantages of our approach are related to the analysis of the integrals of motion of the SODE under consideration, also via a sort of *second quantized* approach, and to the study of the symmetries. Also, our method gives rise quite

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naturally to a perturbation scheme which produces always the solution of the SODE to the wanted order after some purely algebraic computations.

The paper is organized as follows:

in the next section we introduce our strategy and discuss some general results;

in Section 3 we consider the problem of the integrals of motion and analyze in some details a number of examples;

in Section 4 we consider all the mathematical details, which are crucial in order to make the approach rigorous, and then we discuss some preliminary results on symmetries and some future projects.

## 2. THE METHOD

Let us consider the following autonomous SODE in normal form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \ldots, x_N) \\ \dot{x}_2 = f_2(x_1, x_2, \ldots, x_N) \\ \qquad \cdots \\ \qquad \cdots \\ \dot{x}_N = f_N(x_1, x_2, \ldots, x_N) \end{cases} \tag{2.1}$$

with the following initial conditions: $x_j(0) = x_j^o$, for $j = 1, 2, \ldots, N$. Throughout this paper we will suppose that the functions $f_i$ above are such that the solution exists unique. Conditions for that are very well known in the standard literature of ordinary differential equations and will not be discussed here. The only condition which we want to state explicitly is that all the functions $f_j$ are real (this is not essential!) and holomorphic, so that they can be expanded as power series which are uniformly convergent inside a certain polydisk.

This system will be solved using a suggestion coming from quantum mechanics: given a quantum mechanical system $\mathcal{S}$ and the related set of observables $O_{\mathcal{S}}$, that is the set of all the (self-adjoint) bounded (or, more often, unbounded) operators *related* to $\mathcal{S}$, it is well known that the Heisenberg evolution of any $X \in O_{\mathcal{S}}$ satisfies the Heisenberg operator equation of motion

$$\frac{dX(t)}{dt} = i[H, X(t)]. \tag{2.2}$$

Here $H$ is the hamiltonian of the system, which is a self-adjoint densely defined operator acting on the Hilbert space of the theory $\mathcal{H}$, which represents the energy of $\mathcal{S}$, while $[A, B] = AB - BA$ is the commutator between $A, B \in O_{\mathcal{S}}$. All the details of this approach can be found in (Cohen-Tannoudji *et al.*, 1977 Merzbacher, 1970; Schiff, 1968). A formal solution of (2.2), if $H$ does not depend explicitly on time, is $X(t) = e^{iHt} X_0 e^{-iHt}$, where $X_0$ is the initial value of $X(t)$. We say that $e^{iHt} X_0 e^{-iHt}$ is *formal* because this quantity can be defined as a (norm convergent) infinite series, $\sum_{k=0}^{\infty} \frac{(it)^k}{k!}[H, X_0]_k$, only if $H$ and $X$ are bounded while more sophisticated techniques coming from functional analysis and operator theory are required when, e.g., $H$ is unbounded. Here $[A, B]_k$ is the multiple commutator defined recursively as $[A, B]_0 = B$, $[A, B]_k = [A, [A, B]_{k-1}]$. We refer to (Bratteli and Robinson, 1989a,b Reed and Simon, 1980) for some introductory results on this aspect of quantum dynamics and to (Antoine *et al.*, 1999; Bagarello and Trapani, 2002) for some recent results.

We will postpone all the mathematical aspects concerning our approach to the last section, in which we will show how all the steps we are going to propose here can be made rigorous in an operator algebraic framework. In this section and in the following one we will avoid dealing with all these details in order to make more transparent our procedure.

Let us go back to system (2.1). What we have in mind is looking for a (formal) solution of this system of the following form:

$$x_j(t) = e^{iHt} x_j^o e^{-iHt}, \quad j = 1, 2, \ldots, N, \tag{2.3}$$

for some self-adjoint operator $H$. Since the system (2.1) is a classical one, all the variables $x_j$ are classical quantities, so that they must commute among them:

$$[x_j, x_k] = 0 \quad j, k = 1, 2, \ldots, N. \tag{2.4}$$

This means that $[\varphi(x_1, \ldots, x_N), \Phi(x_1, \ldots, x_N)] = 0$ for any two functions $\varphi$ and $\Phi$ depending on $x_j$. For this reason, if $H$ would only depend on $x_j$, we could only get a trivial dynamical behavior since, from (2.3), we get

$$x_j(t) = e^{iHt} x_j^o e^{-iHt} = x_j^o e^{iHt} e^{-iHt} = x_j^o$$

Therefore, such an $H$ can only describes a static behavior. In order to describe some different time evolution a natural possibility is to *double* the space of the variables in the following way: to each variable $x_j^o$ we associate a *canonical conjugate* momentum $p_j$ such that

$$\left[x_j^o, p_k\right] = i\delta_{j,k} \mathbb{1} \quad j, k = 1, 2, \ldots, N. \tag{2.5}$$

This means that the initial position $x_j^o$ must now be considered as an operator acting on some Hilbert space $\mathcal{H}$ (see Section IV), $p_j$ is another operator acting again on $\mathcal{H}$ which satisfies $[x_j^o, p_j] = i\mathbb{1}$ and commutes with all the other position operators $x_k^o$, $k \neq j$. The commutation rules (2.5) must be considered together with the following ones

$$\left[x_j^o, x_k^o\right] = [p_j, p_k] = 0 \quad j, k = 1, 2, \ldots, N, \tag{2.6}$$

which show that variables labelled by different indices are mutually independent. Standard results in QM show that, for any differentiable function $\varphi(x_1^o, \ldots, x_N^o)$, we have

$$\left[p_j, \varphi\left(x_1^o, \ldots, x_N^o\right)\right] = -i\frac{\partial\varphi}{\partial x_j^o}, \quad j = 1, \ldots, N. \tag{2.7}$$

This follows from the fact that, in the so-called $x$-representation of QM, the momentum operator $p_j$ has the following *representation*: $p_j = -i\frac{\partial}{\partial x_j^o}$, while $x_j^o$ behaves simply as a multiplication operator. It is also relevant here to consider functions of the momenta and their commutators with the position operators. We have, for any differentiable function $\hat{\varphi}(p_1, \ldots, p_N)$,

$$\left[x_j^o, \hat{\varphi}(p_1, \ldots, p_N)\right] = i\frac{\partial \hat{\varphi}}{\partial p_j}, \quad j = 1, \ldots, N. \tag{2.8}$$

This is a consequence of the fact that, in the $p$-representation of QM, the position operator $x_j^o$ has the following representation: $x_j^o = i\frac{\partial}{\partial p_j}$, while $p_j$ is now simply a multiplication operator. The $x$ and $p$ representations are unitarily equivalent, so that choosing one instead of the other is merely a fact of opportunity, (Cohen-Tannoudji *et al.*, 1977 Merzbacher, 1970; Schiff, 1968).

Let us now define the following operator:

$$H(\vec{f}_0) = \frac{1}{2}\sum_{j=1}^{N}\{p_j f_j(x_1^o, x_2^o, \ldots, x_N^o) + f_j(x_1^o, x_2^o, \ldots, x_N^o)p_j\}, \tag{2.9}$$

where we have introduced $\vec{f}_0 = (f_1(\vec{X}^o), f_2(\vec{X}^o), \ldots, f_N(\vec{X}^o))$, $\vec{X}^o = (x_1^o, x_2^o, \ldots, x_N^o)$. As we have already said, we will come back on the mathematical properties of all the operators introduced up to now in Section 4. Here we want to remark that calling $H$ *the hamiltonian* of (2.1) can be seen as an abuse of language, since it is absolutely not clear how to distinguish between a kinetic and a potential term in (2.9). Moreover, it is not even clear which is the physical quantum system we are referring to. Finally, as in classical mechanics, usually in QM the kinetic term in an hamiltonian is quadratic in the momenta operators, while the operator $H$ in (2.9) depends on $p_j$ linearly. Nevertheless we will now show that $H$ describes the dynamical behavior of (2.1) exactly in the same way in which an ordinary hamiltonian does in QM. To show this we compute the following time derivative:

$$\begin{aligned}
\frac{d}{dt}\left(e^{iHt}x_j^o e^{-iHt}\right) &= ie^{iHt}\left[H, x_j^o\right]e^{-iHt} = \frac{i}{2}e^{iHt}\big[p_j f_j\left(x_1^o, x_2^o, \ldots, x_N^o\right) \\
&\quad + f_j\left(x_1^o, x_2^o, \ldots, x_N^o\right)p_j, x_j^o\big]e^{-iHt} \\
&= \frac{i}{2}e^{iHt}\big(\left[p_j, x_j^o\right]f_j\left(x_1^o, x_2^o, \ldots, x_N^o\right) \\
&\quad + f_j\left(x_1^o, x_2^o, \ldots, x_N^o\right)\left[p_j, x_j^o\right]\big)e^{-iHt} \\
&= e^{iHt}f_j\left(x_1^o, x_2^o, \ldots, x_N^o\right)e^{-iHt},
\end{aligned}$$

where we have used (2.5), (2.6) and the usual properties of the commutators like, for instance, the following equality: $[AB, C] = [A, C]B + A[B, C]$. Using now the properties of the unitary operators $e^{\pm iHt}$ we deduce easily that, for instance,

$e^{iHt}x_j^o(x_k^o)^2e^{-iHt} = (e^{iHt}x_j^oe^{-iHt})(e^{iHt}x_k^oe^{-iHt})^2$, so that, because of the analyticity of $f_j$, the following crucial equality holds:

$$e^{iHt}f_j\big(x_1^o, x_2^o, \dots, x_N^o\big)e^{-iHt} = f_j\big(e^{iHt}x_1^oe^{-iHt}, e^{iHt}x_2^oe^{-iHt}, \dots, e^{iHt}x_N^oe^{-iHt}\big).$$

We can conclude, therefore, that $\{e^{iHt}x_j^oe^{-iHt},\ j = 1, 2, \dots, N\}$ satisfies (as operators) system (2.1) with the same initial conditions as above. Due to the uniqueness of the solution, therefore, $x_j(t) = e^{iHt}x_j^oe^{-iHt}$ is exactly the solution we were looking for. We will discuss in Section 4 in which sense our operatorial solution coincides with the classical solution of (2.1).

Using now some algebra of the commutators we will recover the following well known classical result

$$\vec{X}(t) = (x_1(t), x_2(t), \dots, x_N(t)) = e^{iHt}\vec{X}^oe^{-iHt} = e^{t\vec{f}_0\cdot\vec{\nabla}_0}\vec{X}^o, \tag{2.10}$$

where $\vec{\nabla}_0 = (\frac{\partial}{\partial x_1^o}, \frac{\partial}{\partial x_2^o}, \dots \frac{\partial}{\partial x_N^o})$.

For that we first need the following result

$$\big[H, h\big(x_1^o, \dots, x_N^o\big)\big]_k = (-i)^k(\vec{f}_0\cdot\vec{\nabla}_0)^k h\big(x_1^o, \dots, x_N^o\big), \quad k = 0, 1, 2, \dots, \tag{2.11}$$

where $h(x_1^o, \dots, x_N^o)$ is any holomorphic function and $H$ the operator in (2.9). This result is proved by induction: the statement for $k = 0$ is trivially true just because of the definition of multiple commutator.

For $k = 1$ the result follows from the commutation rules in (2.5), (2.6) and (2.7).

Finally, using the hypothesis of induction, we get:

$$\begin{aligned}[H, h]_{k+1} &= [H, [H, h]_k] = (-i)^k[H, (\vec{f}_0\cdot\vec{\nabla}_0)^k h] \\ &= (-i)^k\big((-i)(\vec{f}_0\cdot\vec{\nabla}_0)[(\vec{f}_0\cdot\vec{\nabla}_0)^k h]\big) = (-i)^{k+1}(\vec{f}_0\cdot\vec{\nabla}_0)^{k+1}h.\end{aligned}$$

Now, since

$$x_j(t) = e^{iHt}x_j^oe^{-iHt} = \sum_{k=0}^{\infty}\frac{(it)^k}{k!}\big[H, x_j^o\big]_k = x_j^o + \sum_{k=1}^{\infty}\frac{(it)^k}{k!}\big[H, x_j^o\big]_k,$$

and since, for $k \geq 1$, $[H, x_j^o]_k = (-i)^k(\vec{f}_0\cdot\vec{\nabla}_0)^{k-1}f_j(\vec{X}^o)$ (this result can also be proved using induction together with (2.11)), we find:

$$x_j(t) = x_j^o + \sum_{k=1}^{\infty}\frac{(t)^k}{k!}(\vec{f}_0\cdot\vec{\nabla}_0)^{k-1}f_j(\vec{X}^o).$$

It is easy to prove now that $(\vec{f}_0\cdot\vec{\nabla}_0)^{-1}f_j(\vec{X}^o) = x_j^o$. In fact, calling $\varphi_j(\vec{X}^o) = (\vec{f}_0\cdot\vec{\nabla}_0)^{-1}f_j(\vec{X}^o)$, the equation $(\vec{f}_0\cdot\vec{\nabla}_0)\varphi_j(\vec{X}^o) = f_j(\vec{X}^o)$ is solved by $\varphi_j(\vec{X}^o) = x_j^o$. Of course, this does not mean that $(\vec{f}_0\cdot\vec{\nabla}_0)^{-1}$ exists as an everywhere defined operator, but only that it can be defined on $f_j$, which is surely enough for our present purposes. Now we have:

$$x_j(t) = x_j^o + \sum_{k=1}^{\infty} \frac{(t)^k}{k!} (\vec{f}_0 \cdot \vec{\nabla}_0)^k ((\vec{f}_0 \cdot \vec{\nabla}_0)^{-1} f_j(\vec{X}^o))$$

$$= x_j^o + \sum_{k=1}^{\infty} \frac{(t)^k}{k!} (\vec{f}_0 \cdot \vec{\nabla}_0)^k x_j^o = \sum_{k=0}^{\infty} \frac{(t)^k}{k!} (\vec{f}_0 \cdot \vec{\nabla}_0)^k x_j^o = e^{t(\vec{f}_0 \cdot \vec{\nabla}_0)} x_j^o,$$

as we had to prove.

*Remark 2.1.* Notice that in $e^{t\vec{f}_0\cdot\vec{\nabla}_0}\vec{X}^o$ there is no longer reference to the conjugate momenta: this formula looks as a classical one, and, in fact, it is well known in the literature, (Grobner, 1973).

The two expressions for $\vec{X}(t)$ in (2.10) can be interpretated in a peculiar way, which has again an analogous in QM. In fact considering $\vec{X}(t) = e^{iHt}\vec{X}^o e^{-iHt}$, $H$ appears to be the generator of the one-parameter group $U_t : \vec{X}^o \to \vec{X}(t)$ which describes the time evolution of the system. Analogously, $\vec{X}(t) = e^{t\vec{f}_0\cdot\vec{\nabla}_0}\vec{X}^o$ shows that the same time evolution can be described by a semigroup whose generator is $\vec{f}_0 \cdot \vec{\nabla}_0$. This fact, which might appear a bit confusing, is indeed easy to understand: the price we have to pay to move from a dynamical semigroup to a dynamical group, which is by far easier to be handled, is that we have to double the number of variables involved in the description of the system: from $\vec{X}$ to $(\vec{X}, \vec{P})$, where $\vec{P} = (p_1, p_2, \ldots, p_N)$. This fact reflects surprisingly well what happens in the description of any quantum open system $\mathcal{I}$: in fact, suppose that $\mathcal{I}$ is a quantum system interacting with a background $\mathcal{B}$. It is well known, see the standard textbook (Davies, 1976), that the effect of this interaction can be taken into account by using a one-parameter semigroup which describes the dynamical behavior of $\mathcal{I}$. This is because the energy of $\mathcal{I}$ is not preserved during the time evolution (since there is an exchange of energy with $\mathcal{B}$). However, if we *enlarge the system*, considering $\mathcal{I} + \mathcal{B}$ as a whole, it is clear that the situation changes: the energy of this composite system is conserved, and, for this reason, the dynamics can be described by a one-parameter group of transformations. To achieve this result, however, we have to add all the variables related to the background $\mathcal{B}$ to the set of the physical observables. In a sense, this is what we have done in our procedure here: we have introduced a sort of background whose variables, $p_j$, have to be considered together with the original variables of the system, $x_j$. This is the way in which we can get an unitary time evolution.

We use now our approach to prove the so called *exchange theorem*, (Grobner, 1973), restricting to $N = 2$ in (2.1) to simplify the notation: this theorem states essentially that, given the holomorphic functions $\vec{f} = (f_1, f_2)$ and $\varphi(x, y)$, then

$$e^{t\vec{f}\cdot\vec{\nabla}}\varphi(x, y) = \varphi(e^{t\vec{f}\cdot\vec{\nabla}}x, e^{t\vec{f}\cdot\vec{\nabla}}y). \tag{2.12}$$

It is absolutely trivial to prove the *unitary version* of this theorem, as we have already seen before:

$$e^{itH}\varphi(x,y)e^{-itH} = \varphi(e^{itH}xe^{-itH}, e^{itH}ye^{-itH}). \tag{2.13}$$

It is just a bit harder to use our approach to prove formula (2.12). Let us define the following functions:

$$\varphi_1(x,y,t) = e^{t\vec{f}\cdot\vec{\nabla}}\varphi(x,y) \quad \text{and} \quad \varphi_2(x,y,t) = e^{itH}\varphi(x,y)e^{-itH}.$$

Incidentally, we notice that $\varphi_2$ only depends on $x$, $y$ and $t$, as can be deduced expanding $\varphi_2$ as a series of multiple commutators and using (2.7) several times. It is easy to check that $\varphi_1(x,y,t)$ and $\varphi_2(x,y,t)$ satisfy the same differential equation

$$\frac{\partial u}{\partial t} = f_1\frac{\partial u}{\partial x} + f_2\frac{\partial u}{\partial y},$$

with the same initial condition $\varphi_1(x,y,0) = \varphi_2(x,y,0) = \varphi(x,y)$. It is well known, (John, 1982), that under the above assumptions on $\vec{f}$ and $\varphi$, the solution of this equation is unique: this means that $\varphi_1(x,y,t) = \varphi_2(x,y,t)$. We have, therefore:

$$\begin{aligned} e^{t\vec{f}\cdot\vec{\nabla}}\varphi(x,y) &= e^{itH}\varphi(x,y)e^{-itH} = \varphi(e^{itH}xe^{-itH}, e^{itH}ye^{-itH}) \\ &= \varphi(e^{t\vec{f}\cdot\vec{\nabla}}x, e^{t\vec{f}\cdot\vec{\nabla}}y), \end{aligned}$$

because of the (2.10).

*Remark 2.2.* As in the usual approach to SODE, it is quite easy to extend our approach to systems in which the functions $f_j$ in (2.1) depend explicitly on time. The trick is the usual one: a system like this

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, t) \\ \dot{x}_2 = f_2(x_1, x_2, t) \end{cases} \tag{2.14}$$

can be rewritten in the following way simply by adding another variable and an extra equation for this new variable:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = 1. \end{cases} \tag{2.15}$$

This system is again of the form (2.1), with $N = 3$ and $f_3(x_1, x_2, x_3) = 1$.

## 3. INTEGRALS OF MOTION

For our original system (2.1), an integral of motion is a function $I$ of all the $x_j$ such that, calling $x_j(t)$, $j = 1, 2, \ldots, N$, the unique solution of (2.1), we have

$I(x_1(t), \ldots, x_N(t)) = I_0$, constant in time. The relevance of such an integral of motion in the analysis of (2.1) is very well known. What is interesting here is consider this notion in our perspective. First of all, we have to answer an obvious question: what do we have to call, now, integral of motion? This question arises because of the doubling of variables which we have discussed in the previous section. So we introduce the following different definitions:

(a) we still call *integral of motion* (IoM) of system (2.1) any holomorphic function $I$ depending only on the $x_j$ such that $I(x_1(t), \ldots, x_N(t)) = I_0, \forall t$;
(b) then we call *extended integral of motion* (EIoM) of system (2.1) any holomorphic function $J$ depending on the $x_j$, $p_k$ such that $J(x_1(t), \ldots, x_N(t), p_1(t), \ldots, p_N(t)) = J_0$, $\forall t$, where $x_j(t) = e^{iHt} x_j^o e^{-iHt}$ and $p_j(t) = e^{iHt} p_j e^{-iHt}$, $j = 1, 2, \ldots, N$.

Therefore, what we call an IoM is also an integral of motion in the standard sense, while an EIoM is intrinsically related to our approach. Also, it is clear from these definitions that any IoM is a particular EIoM. For this reason, whenever we will need to be general, we will consider EIoM.

In this section we will discuss some strategies to find these integrals of motion (extended or not) and some examples. We will also show that QM suggests an interesting strategy, called *second quantization*, which can be quite useful in finding constants of motion using simple energetic considerations for a fictitious system of quantum particles.

The first result follows from our unitary representation of the time evolution for (2.1): an holomorphic function $J$, depending on the $x_j^o$, $p_k$, is an EIoM if, and only if,

$$\left[H, J\left(x_1^o, \ldots, x_N^o, p_1, \ldots, p_N\right)\right] = 0 \tag{3.1}$$

To prove this statement we recall that, since $J$ is holomorphic, we have

$$\begin{aligned} &J(x_1(t), \ldots, x_N(t), p_1(t), \ldots, p_N(t)) \\ &\quad = (e^{iHt} x_1^o e^{-iHt}, \ldots, e^{iHt} x_N^o e^{-iHt}, e^{iHt} p_1 e^{-iHt}, \ldots, e^{iHt} p_N e^{-iHt}) \\ &\quad = e^{iHt} J\left(x_1^o, \ldots, x_N^o, p_1, \ldots, p_N\right) e^{iHt} \\ &\quad = J\left(x_1^o, \ldots, x_N^o, p_1, \ldots, p_N\right), \end{aligned}$$

because of (3.1).

Therefore the problem of finding EIoM is solved once we can find operators which commute with the hamiltonian $H$ in (2.9). Of course, $H$ is by itself an EIoM in our sense, since it depends on $p_j$ and $x_j^o$ and since $[H, H] = 0$. This implies that any SODE admits at least an EIoM, which is the operator $H$. It is interesting, of course, to discuss here the existence of other EIoM.

Let us see first of all what this commutativity condition becomes for ordinary IoM. In this case we need the commutation rule in (2.11) for $k = 1$, which gives the following equivalence statement:

$$\left[H, J\left(x_1^o, \dots, x_N^o\right)\right] = 0 \Longleftrightarrow (\vec{f}_0 \cdot \vec{\nabla}_0) J\left(x_1^o, \dots, x_N^o\right) = 0, \tag{3.2}$$

which is exactly what we expect since the equation on the right simply states that $\frac{d}{dt} J(x_1(t), \dots, x_N(t)) = 0$. So, at least as far as IoM are concerned, our approach produces the same results as the standard procedure, even if may be easier to find operators commuting with $H$ instead of solving the partial differential equation in (3.2). We will see examples of this fact in Examples 2 and 4 below and when considering the second quantization approach.

Quite different is the situation for EIoM. In order to avoid useless complications, we will consider from now on $N = 2$ in (2.1), since this constraint does not limit the validity of our results and makes all the proofs and statements more readable. Under this assumption, we now prove the following commutation relations, which will be used later in the paper:

$$[p_j, H(\vec{f}_0)]_n = (-i)^n H\left(\vec{f}_{0,x_j^o}^{(n)}\right), \quad n = 0, 1, 2, \dots \tag{3.3}$$

$$\left[H(\vec{f}_0), p_j^n\right] = -\sum_{s=1}^{n} \binom{n}{s} (-i)^s H\left(\vec{f}_{0,x_j^o}^{(s)}\right) p_j^{n-s}, \quad n = 0, 1, 2, \dots \tag{3.4}$$

Moreover, if $\vec{\varphi} = (\varphi_1, \varphi_2)$ and $\vec{\Psi} = (\Psi_1, \Psi_2)$ are regular functions (we will assume that they are at least differentiable two times), then

$$[H(\vec{\varphi}), H(\vec{\Psi})] = H(\vec{\Phi}). \tag{3.5}$$

Here we have introduced the following notation:

$$\vec{f}_{0,x_j^o}^{(n)} = \frac{\partial^n \vec{f}_0}{\partial (x_j^o)^n},$$

and

$$\vec{\Phi} = (\Phi_1, \Phi_2) = i(\vec{\Psi} \cdot (\vec{\nabla}\varphi_1) - \vec{\varphi} \cdot (\vec{\nabla}\Psi_1), \vec{\Psi} \cdot (\vec{\nabla}\varphi_2) - \vec{\varphi} \cdot (\vec{\nabla}\Psi_2)). \tag{3.6}$$

The proof of (3.3) is given by induction. Let us first observe that for $n = 0$ the statement is trivial, while for $n = 1$ it follows from the definition (2.9) of $H$, from the commutation rule (2.7) and from the usual properties of the commutators. For instance we have

$$[p_1, H(\vec{f}_0)] = \frac{1}{2}[p_1, p_1 f_1\left(x_1^o, x_2^o\right) + f_1\left(x_1^o, x_2^o\right) p_1 + p_2 f_2\left(x_1^o, x_2^o\right)$$
$$+ f_2\left(x_1^o, x_2^o\right) p_2] = \frac{1}{2}\left(p_1\left(-i\frac{\partial f_1}{\partial x_1^o}\right) + \left(-i\frac{\partial f_1}{\partial x_1^o}\right) p_1\right.$$

$$+p_2\left(-i\frac{\partial f_2}{\partial x_1^o}\right)+\left(-i\frac{\partial f_2}{\partial x_1^o}\right)p_2\bigg)=-iH\left(\frac{\partial \vec{f}_0}{\partial x_1^o}\right)$$

The second step of induction goes as follows:

$$[p_1, H(\vec{f}_0)]_{n+1}=[p_1,[p_1,H(\vec{f}_0)]_n]=(-i)^n\big[p_1, H\big(\vec{f}^{(n)}_{0,x_j^o}\big)\big]$$
$$=(-i)^n(-i)H\left(\frac{\partial \vec{f}^{(n)}_{0,x_1^o}}{\partial x_1^o}\right)=(-i)^{n+1}H\big(\vec{f}^{(n+1)}_{0,x_j^o}\big)$$

The proof of (3.4) is similar and is left to the reader. Equation (3.5) it is easier to be proved by making use of an equivalent expression for $H$, which follows from (2.9) and (2.7):

$$H(\vec{\varphi})=\vec{\varphi}\cdot\vec{p}-\frac{i}{2}\vec{\nabla}\cdot\vec{\varphi} \tag{3.7}$$

Using this expression we get

$$[H(\vec{\varphi}),H(\vec{\Psi})]=[\vec{\varphi}\cdot\vec{p},\vec{\Psi}\cdot\vec{p}]-\frac{i}{2}[\vec{\varphi}\cdot\vec{p},\vec{\nabla}\cdot\vec{\Psi}]-\frac{i}{2}[\vec{\nabla}\cdot\vec{\varphi},\vec{\Psi}\cdot\vec{p}]$$

since the last contribution $(\frac{i}{2})^2\ [\vec{\nabla}\cdot\vec{\varphi},\vec{\nabla}\cdot\vec{\Phi}]$ is obviously zero. Using (2.7) several times and the definition of $\vec{\Phi}$ we first deduce that $[\vec{\varphi}\cdot\vec{p},\vec{\Psi}\cdot\vec{p}]=\vec{\Phi}\cdot\vec{p}$, while an explicit check, which makes use of equation (2.7) again, shows that

$$-\frac{i}{2}[\vec{\varphi}\cdot\vec{p},\vec{\nabla}\cdot\vec{\Psi}]-\frac{i}{2}[\vec{\nabla}\cdot\vec{\varphi},\vec{\Psi}\cdot\vec{p}]=\frac{i}{2}([\vec{\nabla}\cdot\vec{\Psi},\vec{\varphi}\cdot\vec{p}]-(\vec{\Psi}\leftrightarrow\vec{\varphi}))$$
$$=-\frac{i}{2}\vec{\nabla}\cdot\vec{\Phi}.$$

Therefore formula (3.5) follows.

*Remark 3.1.* Let us take, for instance, $\vec{\Psi}=(1,0)$. Then $H(\vec{\Psi})=p_1$ and (3.5) and 3.6) imply that $[H(\vec{\varphi}),H(\vec{\Psi})]=H(\vec{\Phi})=iH(\frac{\partial\vec{\varphi}}{\partial(x_1)})$. It is clear that (3.3) gives the same result, taking $n=1$ and $\vec{\varphi}=\vec{f}_0$.

Other useful commutation rules are the following

$$[H(\vec{f}),p_1]_n=H\big(\vec{\Phi}^{(n)}_{(x)}\big),\quad [H(\vec{f}),p_2]_n=H\big(\vec{\Phi}^{(n)}_{(y)}\big),\quad n=1,2,3,\ldots \tag{3.8}$$

where we have defined recursively the functions:

$$\begin{cases}\vec{\Phi}^{(0)}_{(x)}=(1,0),\quad \vec{\Phi}^{(0)}_{(y)}=(0,1)\\ \vec{\Phi}^{(n)}_{(x)}=i(\vec{\Phi}^{(n-1)}_{(x)}\cdot(\vec{\nabla}f_1)-\vec{f}\cdot(\vec{\nabla}\vec{\Phi}^{(n-1)}_{(x),1}),\vec{\Phi}^{(n-1)}_{(x)}\cdot(\vec{\nabla}f_2)-\vec{f}\cdot(\vec{\nabla}\vec{\Phi}^{(n-1)}_{(x),2}))\\ \vec{\Phi}^{(n)}_{(y)}=i(\vec{\Phi}^{(n-1)}_{(y)}\cdot(\vec{\nabla}f_1)-\vec{f}\cdot(\vec{\nabla}\vec{\Phi}^{(n-1)}_{(y),1}),\vec{\Phi}^{(n-1)}_{(y)}\cdot(\vec{\nabla}f_2)-\vec{f}\cdot(\vec{\nabla}\vec{\Phi}^{(n-1)}_{(y),2}))\end{cases} \tag{3.9}$$

These commutation rules are proved simply iterating the proof of equation (3.5) with a proper choice of the function $\vec{\Psi}$ at each step of the iteration (for instance, $\vec{\Psi}$ must be taken equal to $(1, 0)$ for the first step to prove the first commutation rule in (3.8), while $\vec{\Psi} = (0, 1)$ for the second one).

We have now all the ingredients to find the time evolution for $p_j$. We have, recalling that $p_1 = H(\vec{\Phi}^{(0)}_{(x)})$,

$$p_1(t) = e^{iHt} p_1 e^{-iHt} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} [H(\vec{f}_0), p_1]_k$$
$$= p_1 + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} H\big(\vec{\Phi}^{(k)}_{(x)}\big) = H\left(\sum_{k=0}^{\infty} \frac{(it)^k}{k!} \vec{\Phi}^{(k)}_{(x)}\right) \tag{3.10}$$

and, analogously,

$$p_2(t) = e^{iHt} p_2 e^{-iHt} = H\left(\sum_{k=0}^{\infty} \frac{(it)^k}{k!} \vec{\Phi}^{(k)}_{(y)}\right). \tag{3.11}$$

Of course, the problem of the convergence of the series has to be considered. We have no general result for this, because of the rather involved expression for $\vec{\Phi}^{(k)}_{(j)}$, $j = x, y$, which makes quite difficult to find some a priori estimates. Of course, there is no problem for $t = 0$ since both the series above reduce to a single contribution. Also, in almost all the examples we will consider, the series both converge explicitly, because of the following general result: if $\vec{\Phi}^{(k)}_{(j)} = \vec{0}$, $j = x$, $j = y$ for a certain $k$, then $\vec{\Phi}^{(l)}_{(j)} = \vec{0}$ for all $l \geq k$. This is a trivial consequence of the definitions (3.9).

Before giving some general statement on the EIoM, we list below some particular result, which can be easily deduced from our previous results:

- $p_1(t)$ is an EIoM, that is $p_1(t) = p_1$, if and only if $\frac{\partial \vec{f}_0}{\partial x_1^o} = \vec{0}$. The easiest way to prove this claim is by considering the commutation rule in (3.3) for $n = 1$ and the definition (2.9). This can also be seen because, if $\vec{f}_0$ does not depend on $x_1^o$, it is clear that $H$ does not depend on $x_1^o$, too. Therefore $[H, p_1] = 0$, and, as a consequence, $p_1(t)$ is constant in $t$.
- For the same reason $p_2(t)$ is an EIoM, if and only if $\frac{\partial \vec{f}_0}{\partial x_2^o} = \vec{0}$ (which implies that $H$ does not depend on $x_2^o$). Of course $\alpha p_1(t) + \beta p_2(t)$ is an EIoM if and only if $\alpha \frac{\partial \vec{f}_0}{\partial x_1^o} + \beta \frac{\partial \vec{f}_0}{\partial x_2^o} = \vec{0}$.
- using (3.4) we can prove that $\alpha p_1^2(t) + \beta p_2^2(t)$ is an EIoM if and only if all the following equalities hold: $\frac{\partial f_1}{\partial x_1^o} = \frac{\partial f_2}{\partial x_2^o} = 0$ and $\alpha \frac{\partial f_2}{\partial x_1^o} + \beta \frac{\partial f_1}{\partial x_2^o} = 0$. These conditions are satisfied only if $f_1(x, y) = k_1 y + m_1$ and $f_2(x, y) = k_2 y + m_2$, with $\alpha k_2 + \beta k_1 = 0$. This shows that there exist very few SODE

(2.1), with $N = 2$, for which $\alpha p_1^2(t) + \beta p_2^2(t)$ turns out to be an EIoM. It is interesting to notice, however, that a very important example satisfies these conditions. This example is the (classical) harmonic oscillator. We will consider this example in details in the following.

- there is no non-trivial SODE (2.1) with $N = 2$ (i.e., with $\vec{f}$ not constant) which admits $\alpha p_1^n(t) + \beta p_2^n(t)$ as an EIoM for $n \geq 3$. This can be easily shown to be a consequence of the fact that $H$ is linear in the momentum operators.
- We end this list with the following claim, which is again a consequence of (3.4): $\alpha p_1^2(t) + \beta p_2^2(t) + \gamma p_1(t)p_2(t)$ is an EIoM of (2.1) with $N = 2$ if all the following equalities are satisfied: $2\alpha \frac{\partial f_1}{\partial x_1^o} + \gamma \frac{\partial f_1}{\partial x_2^o} = 2\beta \frac{\partial f_2}{\partial x_2^o} + \gamma \frac{\partial f_2}{\partial x_1^o} = \vec{\nabla}_0 \cdot \vec{f}_0 = 0$, and $(\alpha \frac{\partial^2}{\partial (x_1^o)^2} + \beta \frac{\partial^2}{\partial (x_2^o)^2} + \gamma \frac{\partial^2}{\partial x_1^o x_2^o})\vec{f}(x_1^o, x_2^o) = \vec{0}$.

Now we deduce a more general result, which extends all the above particular situations, but which is also harder to be applied. The starting point is that, in order for a certain function $A(p_1(t), p_2(t))$ to be an EIoM, the function $B(t) = A(p_1(t), p_2(t)) - A(p_1, p_2)$ must be zero for all values of $t$. In particular, since $B(0) = 0$, it is enough to require that $\dot{B}(t) = 0$, for all $t$. Using (3.10) and (3.11) we conclude that $A$ is an EIoM whenever the following equation is verified:

$$\frac{\partial A}{\partial p_1} H \left( \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \vec{\Phi}_{(x)}^{(k+1)} \right) + \frac{\partial A}{\partial p_2} H \left( \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \vec{\Phi}_{(y)}^{(k+1)} \right) = 0 \tag{3.12}$$

A function $A(p_1(t), p_2(t))$ satisfying this condition can be easily found for a wide class of SODE. Let us consider the following system:

$$\begin{cases} \dot{x}_1 = f_1(ax_1 + bx_2) \\ \dot{x}_2 = f_2(ax_1 + bx_2) \end{cases} \tag{3.13}$$

with $x_1(0) = x_1^o$, $x_2(0) = x_2^o$ and $a, b$ both different from zero. We will show now that equation (3.12) implies that any function $A(p_1(t), p_2(t)) = A(\frac{1}{a} p_1(t) - \frac{1}{b} p_2(t))$ is an EIoM.

To prove this claim, first of all we observe that such an $A$ must satisfy the following equation: $a \frac{\partial A}{\partial p_1} = -b \frac{\partial A}{\partial p_2}$. Therefore equation (3.12) becomes, using the linearity of $H$ in the $p_j$'s,

$$-\frac{1}{a} \frac{\partial A}{\partial p_2} H \left( \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left( b\vec{\Phi}_{(x)}^{(k+1)} - a\vec{\Phi}_{(y)}^{(k+1)} \right) \right) = 0.$$

To avoid trivial solution ($A$ independent of $p_j$), we need to require $\frac{\partial A}{\partial p_2} \neq 0$. This implies that the equation above can be satisfied if, and only if,

$$\sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left( b\vec{\Phi}^{(k+1)}_{(x)} - a\vec{\Phi}^{(k+1)}_{(y)} \right) = \vec{0} \tag{3.14}$$

In order to check that this equality is satisfied for the system in (3.13) we first notice that, whenever $b\vec{\Phi}^{(k)}_{(x)} - a\vec{\Phi}^{(k)}_{(y)} = \vec{0}$, then $b\vec{\Phi}^{(j)}_{(x)} - a\vec{\Phi}^{(j)}_{(y)} = \vec{0}$, for any $j \geq k$. In fact we have:

$$\begin{aligned} b\vec{\Phi}^{(k+1)}_{(x)} - a\vec{\Phi}^{(k+1)}_{(y)} &= ib\left( \vec{\Phi}^{(k)}_{(x)} \cdot (\vec{\nabla} f_1) - \vec{f} \cdot \left(\vec{\nabla}\vec{\Phi}^{(k)}_{(x),1}\right), \vec{\Phi}^{(k)}_{(x)} \cdot (\vec{\nabla} f_2) \right. \\ &\quad \left. - \vec{f} \cdot \left(\vec{\nabla}\vec{\Phi}^{(k)}_{(x),2}\right)\right) \\ &\quad - ia\left( \vec{\Phi}^{(k)}_{(y)} \cdot (\vec{\nabla} f_1) - \vec{f} \cdot \left(\vec{\nabla}\vec{\Phi}^{(k)}_{(y),1}\right), \vec{\Phi}^{(k)}_{(y)} \cdot (\vec{\nabla} f_2) \right. \\ &\quad \left. - \vec{f} \cdot \left(\vec{\nabla}\vec{\Phi}^{(k)}_{(y),2}\right)\right) \\ &= i\Big\{ \overbrace{\left(b\vec{\Phi}^{(k)}_{(x)} - a\vec{\Phi}^{(k)}_{(y)}\right)}^{=\vec{0}} \cdot (\vec{\nabla} f_1) - \vec{f}_0 \cdot \Big(\vec{\nabla} \overbrace{\left(b\vec{\Phi}^{(k)}_{(x),1} - a\vec{\Phi}^{(k)}_{(y),1}\right)}^{=0}\Big), \\ &\quad \underbrace{\left(b\vec{\Phi}^{(k)}_{(x)} - a\vec{\Phi}^{(k)}_{(y)}\right)}_{=\vec{0}} \cdot (\vec{\nabla} f_2) - \vec{f}_0 \cdot \Big(\vec{\nabla} \underbrace{\left(b\vec{\Phi}^{(k)}_{(x),2} - a\vec{\Phi}^{(k)}_{(y),2}\right)}_{=0}\Big)\Big\} = \vec{0} \end{aligned}$$

It is straightforward to extend the proof to larger values of $j$.

Going back to equation (3.14), the proof of this equality is now reduced to the proof of the following identity: $b\vec{\Phi}^{(1)}_{(x)} - a\vec{\Phi}^{(1)}_{(y)} = \vec{0}$. But this is an easy consequence of the equalities $\vec{\Phi}^{(1)}_{(x)} = i(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1})$, $\vec{\Phi}^{(1)}_{(y)} = i(\frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_2})$ and from the dependence of $f_j$ only on the combination $ax_1 + bx_2$.

As a concrete example of this result, we now consider the following SODE

$$\begin{cases} \dot{x}_1 = e^{x_1 - x_2} \\ \dot{x}_2 = \cos(x_1 - x_2) + 1, \end{cases}$$

which is of the form (3.13) with $a = -b = 1$. Its Hamiltonian is

$$\begin{aligned} H = \frac{1}{2}\Big(& p_1 e^{x_1^o - x_2^o} + e^{x_1^o - x_2^o} p_1 + p_2\big(\cos(x_1^o - x_2^o) + 1\big) \\ &+ \big(\cos(x_1^o - x_2^o) + 1\big) p_2\Big). \end{aligned}$$

We will now compute explicitly $[H, p_1 + p_2]$ to check that $A(p_1(t), p_2(t)) = p_1(t) + p_2(t)$ is a constant of motion, as can be deduced from the above procedure.

Using (2.7) we find that

$$[e^{x_1^o - x_2^o}, p_1] = i e^{x_1^o - x_2^o}, \quad [e^{x_1^o - x_2^o}, p_2] = -i e^{x_1^o - x_2^o},$$

and

$$\left[\cos\left(x_1^o - x_2^o\right) + 1, p_1\right] = -i \sin\left(x_1^o - x_2^o\right), \quad \left[\cos\left(x_1^o - x_2^o\right) + 1, p_2\right] = i \sin\left(x_1^o - x_2^o\right),$$

so that

$$\begin{aligned}[2H, p_1 + p_2] = {} & p_1[e^{x_1^o - x_2^o}, p_1 + p_2] + [e^{x_1^o - x_2^o}, p_1 + p_2]p_1 \\ & + p_2\left[\cos\left(x_1^o - x_2^o\right) + 1, p_1 + p_2\right] + +\left[\cos\left(x_1^o - x_2^o\right)\right. \\ & \left. + 1, p_1 + p_2\right]p_2 = 0,\end{aligned}$$

which is what we had to prove.

In the following part of this section we will analyze in some details few examples.

*Example 3.1.* The harmonic oscillator

We consider first one of the best known system in classical mechanics: the harmonic oscillator. The related SODE is

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x, \end{cases}$$

with $x(0) = x_0$, $y(0) = y_0$. The classical solution is $x(t) = x_0 \cos(\omega t) + \frac{y_0}{\omega} \sin(\omega t)$ and $y(t) = y_0 \cos(\omega t) - x_0 \omega \sin(\omega t)$. As for the classical IoM, this is clearly the energy of the oscillator, $E = y(t)^2 + \omega^2 x(t)^2$. Let us now see what can be deduced using our strategy.

Since $f_1(x, y) = y$, $f_2(x, y) = -\omega^2 x$, the Hamiltonian is $H = p_1 y_0 - \omega^2 p_2 x_0$. The time evolution $x(t)$ and $y(t)$ can be found computing $e^{iHt}(x_0, y_0)e^{-iHt}$ or using the last equality of formula (2.10), $e^{t\vec{f}_0 \cdot \vec{\nabla}_0}(x_0, y_0)$. It is an easy exercise to compute the various commutators in

$$x(t) = x_0 + it[H, x_0] + \frac{(it)^2}{2!}[H, x_0]_2 + \frac{(it)^3}{3!}[H, x_0]_3 + \cdots.$$

For instance $[H, x_0] = [p_1 y_0, x_0] = -i y_0$, $[H, x_0]_2 = -i[H, y_0] = -i[-\omega^2 p_2 x_0, y_0] = \omega^2 x_0$ and so on. When we sum back the series, we get the same classical result. Analogous computations must be performed to find $y(t)$.

We can now compute also the time evolution of the momentum operators, with the same technique ($p_j(t) = e^{iHt} p_j e^{-iHt}$) or by means of (3.10)–(3.11).

For instance we get

$$p_1(t) = e^{iHt} p_1 e^{-iHt} = p_1 + it[H, p_1] + \frac{(it)^2}{2!}[H, p_1]_2 + \frac{(it)^3}{3!}[H, p_1]_3 \cdots .$$
$$= p_1 + t\omega^2 p_2 - \frac{t^2}{2!}\omega^2 p_1 - \frac{t^3}{3!}\omega^4 p_2 + \cdots = p_1 \cos(\omega t) + \omega p_2 \sin(\omega t).$$

Analogously we find $p_2(t) = p_2 \cos(\omega t) - \frac{p_1}{\omega} \sin(\omega t)$. In order to use formula (3.10) we need to compute first $\vec{\Phi}^{(k)}_{(x)}$, by means of the definition in (3.9). It is easy to check that $\vec{\Phi}^{(0)}_{(x)} = (1, 0)$, $\vec{\Phi}^{(1)}_{(x)} = i(0, -\omega^2)$, $\vec{\Phi}^{(2)}_{(x)} = (\omega^2, 0)$, $\vec{\Phi}^{(3)}_{(x)} = -i(0, \omega^4)$, $\vec{\Phi}^{(4)}_{(x)} = (\omega^4, 0)$ and so on, and therefore

$$H\left(\sum_{k=0}^{\infty} \frac{(it)^k}{k!} \vec{\Phi}^{(k)}_{(x)}\right) = H\left((1, 0) + t(0, \omega^2) - \frac{t^2}{2!}(\omega^2, 0) + \frac{(it)^3}{3!}(-i)(0, \omega^4) + \cdots\right)$$

which, computing the infinite sum, gives the same result as above. Analogous steps must be repeated for computing $p_2(t)$.

We now look for EIoM.

Equation (3.2) becomes $y\frac{\partial J_1}{\partial x} - \omega^2 x \frac{\partial J_1}{\partial y} = 0$, which is solved by $J_1(x, y) = y^2 + \omega^2 x^2$, so that we find back the classical result.

Another EIoM is the following: $J_2(p_1(t), p_2(t)) = p_1(t)^2 + \omega^2 p_2(t)^2$. This claim can be proved simply checking that $[H, J_2] = 0$, or noticing that this example fits the conditions given at the third point of the list of particular EIoM given previously. We also can prove this claim with a direct substitution of $p_j(t)$, as computed before: this gives $p_1(t)^2 + \omega^2 p_2(t)^2 = p_1^2 + \omega^2 p_2^2$. Finally, we can also use equation (3.12), and show that $J_2$ is a solution of this equation.

As this example suggests, the relevance of our strategy as far as we are interested to the explicit solution of a given SODE, is strongly related to (a) the possibility of computing easily multiple commutators with a relatively simple operator, $H$, and, (b) to the possibility of summing the infinite series we have obtained in this way. While the first step can always be performed, this summation might not be easy or even possible. However, it is clear that this strategy produces quite naturally a perturbative approach to the dynamical problem: if we have some estimate on the rest of the series, $\Sigma_{k=n+1}^{\infty} \frac{(it)^k}{k!}[H, A]_k$, where $A$ is the operator whose time evolution we are interested in, then we can say exactly in which sense $A(t)$ can be approximated by the finite sum $\Sigma_{k=0}^{n} \frac{(it)^k}{k!}[H, A]_k$. This kind of estimates appear almost everywhere in QM, and for this reason many techniques have been developed during the years in order to obtain some useful bounds, see, e.g., (Bagarello and Trapani, 2002) and references therein.

*Example 3.2.* Euler's equation of the rigid body

This is another well known example arising in classical mechanics. Let $A$, $B$ and $C$ be positive constants. The SODE is

$$\begin{cases} \dot{x} = \frac{B-C}{A} yz \\ \dot{y} = \frac{C-A}{B} xz \\ \dot{z} = \frac{A-B}{C} xy \end{cases}$$

with $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$. As it is well known, these equations are easily solved only for some particular choices of the constants, e.g. for $A = B$. The hamiltonian of the system is $H = \frac{B-C}{A} y_0 z_0 p_1 + \frac{C-A}{B} x_0 z_0 p_2 + \frac{A-B}{C} x_0 y_0 p_3$.

It is easy to build up, now, two IoM using our strategy. The basic ingredient is the commutation relation (2.7), which in particular implies that $[p_1, x_0^2] = -2ix_0$, $[p_2, y_0^2] = -2iy_0$ and $[p_3, z_0^2] = -2iz_0$. This fact, together with the analytical expression of $H$, suggests to consider a generic linear combination of $x_0^2$, $y_0^2$ and $z_0^2$, $J = \alpha x_0^2 + \beta y_0^2 + \gamma z_0^2$, since, with this choice, the commutator $[H, J]$ contains the common factor $x_0 y_0 z_0$ multiplied by a constant. More explicitly we have

$$[H, J] = -2ix_0 y_0 z_0 \left( \frac{B-C}{A} \alpha + \frac{C-A}{B} \beta + \frac{A-B}{C} \gamma \right),$$

which is identically zero if and only if $\frac{B-C}{A} \alpha + \frac{C-A}{B} \beta + \frac{A-B}{C} \gamma = 0$. This last equation is satisfied if $\alpha = A$, $\beta = B$ and $\gamma = C$ or alternatively if $\alpha = A^2$, $\beta = B^2$ and $\gamma = C^2$. We have recovered in this way the classical result which states that both $J_1 = Ax(t)^2 + By(t)^2 + Cz(t)^2$ and $J_2 = A^2 x(t)^2 + B^2 y(t)^2 + C^2 z(t)^2$ are IoM. We see here that the computation of an IoM is reduced to a (nontrivial) algebraic operation: the analysis of the commutant of the operator $H$, i.e. the set of operators which commute with $H$. We refer to (Bratelli and Robinson, 1979) for many mathematical results concerning the theory of commutants.

Other quantities which commute with $H$ can be found for particular choices of the constants: if $A = B$ then $J_3 = z(t)$ is an IoM, which of course implies that $x^2(t) + y^2(t)$ does not depend on time as well; if $A = B = C$ the system trivializes and $J_4 = \alpha x(t) + \beta y(t) + \gamma z(t)$ is an IoM for all values of $\alpha$, $\beta$ and $\gamma$; finally, if $\frac{B-C}{A} + \frac{C-A}{B} = 0$, then $(x_0^2 + y_0^2)^2$ commutes with $H$ and $J_5 = (x(t)^2 + y(t)^2)^2$ is an IoM. It is particularly interesting the situation in which $A = B$: in this case we have $H = \frac{z_0}{2}(y_0 p_1 - x_0 p_2)$, which is essentially proportional to the $z$-component of the angular momentum operator $\vec{L} = \vec{r}_0 \wedge \vec{p}$. It is well known that, in spherical coordinates, $L_z = -i \frac{\partial}{\partial \varphi}$, (Cohen-Tannoudji *et al.*, 1977 Merzbacher, 1970; Schiff, 1968). We are using unities in which $\hbar = 1$. Therefore we have $H = \frac{i}{2} r \cos\theta \frac{\partial}{\partial \varphi}$, which implies clearly that $A(r, \theta, \varphi)$ is an IoM if and only if it is independent of $\varphi$. Then, since $r = \sqrt{x^2 + y^2 + z^2}$ and $\theta = \arccos(z/r)$, it is straightforward to recover the previous result: $z(t)$ and $x^2(t) + y^2(t)$ are both IoM.

As for the EIoM, it is easy to check that, if for instance $A = B$, besides $J_3$, also $J_6 = y(t)p_1(t) - x(t)p_2(t)$ is constant in time, since $[H, y_0p_1 - x_0p_2] = 0$. Notice that $J_6$ has the same formal expression of an angular momentum, and this is not surprising since $A = B$ implies the existence of a symmetry in the $(x, y)$-plane.

*Example 3.3.* Let us consider the following SODE

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = x_1 \end{cases}$$

with $x_j(0) = x_j^o$, $j = 1, 2, 3, 4$. The hamiltonian of the system is $H = p_1x_2^o + p_2x_3^o + p_3x_4^o + p_4x_1^o$, and the time evolution of, say, $x_1$ is obtained with the usual expansion: $x_1(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}[H, x_1^o]_k$. The behavior of these commutators is cyclic: $[H, x_1^o]_0 = x_1^o$, $[H, x_1^o]_1 = -ix_2^o$, $[H, x_1^o]_2 = -x_3^o$, $[H, x_1^o]_3 = ix_4^o$ and, again, $[H, x_1^o]_4 = x_1^o$. The sum of the series for $x_1(t)$ can be computed and we get

$$x_1(t) = \frac{1}{4}\left(x_1^o + x_2^o + x_3^o + x_4^o\right)e^t + \frac{1}{4}\left(x_1^o - x_2^o + x_3^o - x_4^o\right)e^{-t} + \frac{1}{4}\left(x_1^o - ix_2^o - x_3^o + ix_4^o\right)e^{it} + \frac{1}{4}\left(x_1^o + ix_2^o - x_3^o - ix_4^o\right)e^{-it},$$

which coincides with the classical result.

For what concerns the EIoM, it is not difficult to construct an operator $E$ which commutes with $H$. This operator is $E = p_2x_1^o + p_3x_2^o + p_4x_3^o + p_1x_4^o$: $[H, E] = 0$. Notice that $E$ can be obtained from $H$ in the following way: $E = H(x_j^o \leftrightarrow p_j)$.

*Example 3.4.* Lotka-Volterra system

Let us consider the following SODE

$$\begin{cases} \dot{x} = \alpha x - \beta xy = f_1(x, y) \\ \dot{y} = -\gamma y + \beta xy = f_2(x, y), \end{cases}$$

where $\alpha$, $\beta$ and $\gamma$ are real constant. This system is very well known in the biological literature since it describes a two-species system. The analytical solution cannot be obtained but for special values of the constants. Nevertheless, it is well known that $I(x, y) = \beta(x + y) - \gamma \log(x) - \alpha \log(y)$ is an IoM for this system.

This result can be recovered following our strategy: first we remark that the hamiltonian for the SODE above looks like $H = x_0(\alpha - \beta y_0)p_1 + y_0(-\gamma + \beta x_0)p_2 + \frac{i}{2}\beta(y_0 - x_0)$, neglecting an additional term in the definition of $H$, which is irrelevant since it commutes with everything, (Cohen-Tannoudji *et al.*, 1977 Merzbacher, 1970; Schiff, 1968). In order to find an IoM we should be able to find elements in the commutant of $H$. This can be done more conveniently by considering first the change of variable $q_1 = \log x$ and $q_2 = \log y$, which maps the original SODE into

$$\begin{cases} \dot{q}_1 = \alpha - \beta e^{q_2} \\ \dot{q}_2 = -\gamma + \beta e^{q_1}. \end{cases}$$

With the same strategy developed in Section II we introduce two momentum operators canonically conjugate to $q_1^o = \log x_0$ and $q_2^o = \log y_0$: $[q_j^o, P_k] = i\delta_{j,k}\mathcal{I}$, for $j, k = 1, 2$. In these new variables the hamiltonian takes the form $\tilde{H} = P_1(\alpha - \beta e^{q_2^o}) + P_2(-\gamma + \beta e^{q_1^o})$, which can be conveniently written as $\tilde{H} = H_0 + H_1$, where $H_0 = \alpha P_1 - \gamma P_2$ and $H_1 = \beta(P_2 e^{q_1^o} - P_1 e^{q_2^o})$. Because of its analytic expression it is now very easy to find an operator $I_0(q_1^o, q_2^o)$ which commutes with $H_1$: in fact, since (2.7) implies that

$$[H_1, I_0] = -i\beta \left( e^{q_1^o} \frac{\partial I_0}{\partial q_2^o} - e^{q_2^o} \frac{\partial I_0}{\partial q_1^o} \right),$$

it is enough to take $I_0(q_1^o, q_2^o) = a(e^{q_1^o} + e^{q_2^o})$, $a$ any complex number, to conclude that $[H_1, I_0] = 0$. However $I_0$ is not an IoM because we have $[H_0, I_0] = ia(\gamma e^{q_1^o} - \alpha e^{q_2^o}) \neq 0$. Therefore, we can try to add a contribution to $I_0$, $I_1$, in such a way that $I = I_0 + I_1$ is an IoM. In other words, we are trying to find a function $I_1(q_1^o, q_2^o)$ such that

$$[H, I] = ia(\gamma e^{q_1^o} - \alpha e^{q_2^o}) + [H_0, I_1] + [H_1, I_1] = 0.$$

Since $H_0$ is linear in the momentum operators and does not depend on $q_j^o$, it is clear that $J(q_1^o, q_2^o) = \gamma q_1^o + \alpha q_2^o$ commutes with $H_0$: $[H_0, J] = 0$. Furthermore, using (2.7), we find $[H_1, J] = i\beta(\gamma e^{q_1^o} - \alpha e^{q_2^o})$. Therefore, it is enough to take $a = -\beta$ and $I_1 = J$ to conclude that $I(q_1^o, q_2^o) = -\beta(e^{q_1^o} + e^{q_2^o}) + \gamma q_1^o + \alpha q_2^o$ is an IoM. It is also easy to check that, in the original variables, we recover immediately the classical result. Also in this case we have solved the problem of finding an IoM to the analysis of the commutant of the hamiltonian $H$, (Bratteli and Robinson, 1979a,b).

For what concerns the EIoM, we know from the general theory that at least one (proper) EIoM exists for such a system: $H$ itself. Unfortunately, the techniques developed in this section does not allow to find easily other EIoM, but for very special choices of the constants. For instance, if $\beta = 0$ then both $J_1 = y_0 p_2$ and $J_2 = x_0 p_1$ commute with $H$.

However, our procedure produces some interesting hints if we rewrite $H$ as $H = \alpha x_o p_1 - \gamma y_0 p_2 - \beta x_0 y_0 (p_1 - p_2) + \frac{i}{2}\beta(y_0 - x_0)$, we are suggested to consider $A(x(t), y(t)) = x(t) + y(t)$ as a possible IoM, since $x_o + y_0$ surely commutes with the second and the third terms in $H$. However, calling $H_1 = \alpha x_o p_1 - \gamma y_0 p_2$, we find that $[H_1, x_0 + y_0] = i(\gamma y_0 - \alpha x_0)$, so that $A$ cannot be an IoM. However, this result suggests to analyze the case in which $\alpha = -\gamma$. In this case, in fact, it turns out that $[H, x_0 + y_0]_k = (i\gamma)^k (x_0 + y_0)$, so that $x(t) + y(t) = e^{-\gamma t}(x_0 + y_0)$. This result coincides with the classical one, which can be deduced from the system above under the same condition.

Let us now define $x_N(t) = \sum_{k=0}^{N} \frac{(it)^k}{k!}[H, x_0]_k$ and $y_N(t) = \sum_{k=0}^{N} \frac{(it)^k}{k!}[H, y_0]_k$, for some fixed $N \in N$. It is clear that $x_N(t)$ and $y_N(t)$ approximate, in some way, the correct solutions $x(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}[H, x_0]_k$ and $y(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}[H, y_0]_k$. In order to see in which sense $x(t)$ and $y(t)$ are approximated, instead of considering, e.g., $x(t) - x_N(t)$, we will consider $I(x_N(t), y_N(t))$. This is clearly a matter of convenience: while an estimate of $x(t) - x_N(t)$ is hard since we have to estimate the rest of a series, it is much easier to estimate $\Delta_N(t) = I(x_N(t), y_N(t)) - I(x_0, y_0)$. However, this kind of estimate can only give an idea of the validity of our approximation, while a rigorous result can only be obtained considering $x(t) - x_N(t)$. Just as a pedagogical example, we take here only the first non trivial approximation, $N = 1$. Since

$$\begin{cases} x_1(t) = x_0 + it[H, x_0] = x_0(1 - t(\gamma + \beta y_0)), \quad \text{and} \\ y_1(t) = y_0 + it[H, y_0] = y_0(1 - t(\gamma - \beta x_0)), \end{cases}$$

we get $\Delta_1(t) = -t\beta\gamma(x_0 + y_0) + \gamma \log(\frac{1-t(\gamma-\beta x_0)}{1-t(\gamma+\beta y_0)})$. It is not hard to check then that, whenever $t(\gamma + \beta y_0) < 1$, $\Delta_1(t) \leq \frac{t^2\beta\gamma(x_0+y_0)(\gamma+\beta y_0)}{1-t(\gamma+\beta y_0)}$, which is quite small for $t$ small enough. This suggests that, at least for small values of $t$, $x_1(t)$ and $y_1(t)$ are a reasonable approximation of $x(t)$ and $y(t)$. We don't want to go on with this perturbative analysis of the solution of the model, since this is not our major interest, here. More details will be given in a further paper.

We end this section introducing an approach, arising from quantum many body theory, (Cohen-Tannoudji *et al.*, 1977; Merzbacher, 1970; Schiff, 1968), which turns out to be particularly interesting in finding the integrals of motion of a given SODE, because it maps the original system of differential equations in a quantum system of $N$ kinds of bosonic excitations. As before, in order to maintain the notation simple, we fix $N = 2$ in (2.1).

As it is widely discussed in the literature to any conjugate pair $(x, p)$ of self-adjoint (unbounded) operators satisfying $[x, p] = i\mathbb{1}$, it is possible to associate two operators, called the *creation* and *annihilation* operators $a$ and $a^\dagger$ satisfying the *canonical commutation relation* (CCR) $[a, a^\dagger] = \mathbb{1}$, in the following way

$$a = \frac{x + ip}{\sqrt{2}}, \quad a^\dagger = \frac{x - ip}{\sqrt{2}}, \text{ and, vice versa,} \quad x = \frac{a + a^\dagger}{\sqrt{2}}, \quad p = i\frac{a^\dagger - a}{\sqrt{2}}.$$

We refer to (Cohen-Tannoudji *et al.*, 1977; Merzbacher, 1970; Schiff, 1968) for many details concerning the use of these operators, which give rise to what is usually called *second quantization*. Here we just want to remind few facts about these operators. Any time we have a pair of operators satisfying the CCR, we can also define a *number* operator $N = a^\dagger a$ and a set of vectors $\Psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n \Psi_0$, where $\Psi_0$, called the *ground state*, is uniquely defined (up to a phase), by the requirement that $a\Psi_0 = 0$. The following equations hold:

$$\begin{cases} N\Psi_n = n\Psi_n, \forall n \geq 0 \\ N(a\Psi_n) = (n-1)(a\Psi_n), \text{ for } n = 1, 2, \ldots, \\ N(a^\dagger \Psi_n) = (n+1)(a^\dagger \Psi_n), \forall n \geq 0. \end{cases}$$

Because of these equations, we say that $\Psi_n$ is a vector with $n$ bosons, that $a$ annihilates one boson while $a^\dagger$ creates one boson. $N$ is called the number operator just because it counts the bosons of a given vector.

Let us consider now the following SODE, which is clearly associated to an harmonic oscillator with $\omega = 1$, see Example 1:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1. \end{cases}$$

The operator $H$ in (2.9) becomes $H = p_1 x_2^o - x_1^o p_2$. Since we have two pairs of conjugate operators, $(x_1^o, p_1)$ and $(x_2^o, p_2)$, we need to introduce two different pairs of creation and annihilation operators, $(a_1, a_1^\dagger)$ and $(a_2, a_2^\dagger)$, whose definitions are the same for each boson mode. The CCR are now

$$[a_i, a_j^\dagger] = \mathbb{1}\delta_{i,j}, \quad [a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0. \tag{3.15}$$

In terms of these operators we find that $H = i(a_2 a_1^\dagger - a_1 a_2^\dagger)$, which has a clear physical interpretation: the first term annihilates a boson in mode 2 (briefly, a 2-boson) and creates a boson in mode 1 (briefly, a 1-boson). The second term does exactly the opposite. For this reason, it is clear that the total number of the bosons is preserved by $H$, which means that the total number operator $N = a_1^\dagger a_1 + a_2^\dagger a_2$ should commute with $H$. This can be checked explicitly, by means of the (3.15): $[H, N] = 0$. In terms of the original operators $N$ assumes the following expression: $N = \frac{1}{2}((x_1^o)^2 + p_1^2) + \frac{1}{2}((x_2^o)^2 + p_2^2) - \mathbb{1}$, which, of course, must not be confused with the classical result which states that the sum of the kinetic and potential energy is constant for the oscillator. Moreover, it is interesting to notice that, in a certain sense, $H$ and $N$ contain the same dynamical information. To see this, let us first notice that $[N, x_1^o] = -ip_1$, $[N, x_2^o] = -ip_2$, $[N, p_1] = ix_1^o$ and $[N, p_2] = ix_2^o$. Therefore, calling $X(t) = e^{iNt} x_1^o e^{-iNt}$ and $P_X(t) = e^{iNt} p_1 e^{-iNt}$, we deduce that $\dot{X}(t) = P_X(t)$ and $\dot{P}_X(t) = -X(t)$, so that $\ddot{X}(t) = -X(t)$. Analogously, defining $Y(t) = e^{iNt} x_2^o e^{-iNt}$, we find $\ddot{Y}(t) = -Y(t)$. Therefore, since the initial conditions coincide with those for $x_1(t)$ and $x_2(t)$, we deduce that $x_1(t) = X(t)$ and $x_2(t) = Y(t)$. This aspect of our approach, which looks quite appealing to us, should be further investigated since it suggests that, at least sometimes, an operator commuting with $H$ can produce the same dynamical behavior as the one given by $H$.

We want to show now how the example considered above can produce non trivial information in other situations. Again, for pedagogical reasons, we consider a system of only two differential equations, or, in a second quantization language, only two modes of bosons: mode 1 and mode 2. We suppose that the hamiltonian $H$, in terms of the creation and annihilation operators, takes the form $H = (a_1^\dagger)^n a_2 + a_2^\dagger a_1^n$. In the same way as before, the action of $H$ on a vector $\Psi(1)_k \times \Psi(2)_l$ produce a combination of the following vectors $\Psi(1)_{k+n} \times \Psi(2)_{l-1}$ and $\Psi(1)_{k-n} \times \Psi(2)_{l+1}$ (with the agreement that $\Psi(i)_j$ is the zero vector if $j < 0$). It is clear that the same argument of conservation of the total number of bosons cannot be repeated here: a vector with $k + l$ bosons is mapped into a combination of vectors with $k + n + l - 1$ and $k - n + l + 1$ bosons respectively! However, it is also clear that to produce $n$ 1-bosons we have to destroy 1 2-boson. Moreover, in order to produce one 2-boson, we need to destroy $n$ 1-bosons. All that suggests the existence of a conserved operator which is a sort of total energy for the two modes of bosons: $E = a_1^\dagger a_1 + n a_2^\dagger a_2$. This operator looks identical to an hamiltonian of an harmonic oscillator with two modes, the first with frequency $\omega_1 = 1$ and the second with frequency $\omega_2 = n$, (Cohen-Tannoudji *et al.*, 1977; Merzbacher, 1970; Schiff, 1968). Finally, it is straightforward to check explicitly that $[H, E] = 0$, as expected.

These simple examples show how this *second quantization language* may be relevant in the analysis of a given SODE because it transfers the original problem into a totally different settings, where simple "energetic" remarks can be used in the search of EIoM.

## 4. SOME MATHEMATICAL RIGOR AND MORE INFORMATION

We devote this section to a short discussion of some mathematical details concerning the procedure developed in the previous sections.

In our approach the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{4.1}$$

is considered as an operatorial SODE: $x_j$, $f_k$ are operators acting on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\boldsymbol{R}^2)$. For each variable $x_j$ we can introduce a conjugate operator $p_j$, $[x_j, p_j] = i\mathbb{1}$, which also acts on $\mathcal{H}$. It is well known that, because of these commutation rules, $x_j$ and $p_j$ are both (self-adjoint) unbounded operators, so that they cannot be defined on all of $\mathcal{H}$ but only on the dense domain $\mathcal{D} = \mathcal{S}(\boldsymbol{R}^2)$, see e.g. (Reed and Simon, 1980). It is clear that all the powers of the position and momentum operators are self-adjoint and that they map $\mathcal{S}(\boldsymbol{R}^2)$ into itself.

For what concerns $H$, which is the key operator in our strategy, it is possible to show that its domain, $D(H)$, contains $\mathcal{D}$ and that, for all $\varphi, \Psi \in D(H)$ the following equality is satisfied: $< H\varphi, \Psi >=< \varphi, H\Psi >$. This implies that $H$ is symmetric, so that it can be exponentiated using standard techniques of functional analysis, (Reed and Simon, 1980), under few additional assumptions. This is an important feature since the series $\sum_{k=0}^{\infty} \frac{(itH)^k}{k!}$, has no meaning in general when $H$ is not everywhere defined. In this case, however, many other approaches have been considered in the literature to give a rigorous meaning to $e^{iHt}$ and to the (Heisenberg-)time evolution of the operators, $\alpha^t$, see (Bagarello and Trapani, 2002) and (Bagarello *et al.*, 2004) for instance.

Another remark, still related to the definition of $\alpha^t$, is about the rigorous definition of the *commutant of* $H$ and, more generally, the meaning of formulas like $\sum_{k=0}^{\infty} \frac{(it)^k}{k!}[H, x_1^o]_k$: in fact, since $H$ is usually an unbounded operator, as well as $x_1^o$, the commutant of $H$ must be understood in a weak sense, (Bratteli and Robinson, 1979a,b). Also, while $[H, x_1^o]$ may have no meaning, $< \varphi, [H, x_1^o]\Psi >$ is well defined if $\varphi$ and $\Psi$ are both taken in $\mathcal{D}$. Moreover, it is also to be considered the problem of the convergence of the series which define the time evolution of a given operator, but this is a very hard problem and there are few general results on this point in our knowledge but for a general strategy proposed in (Bagarello and Trapani, 2002), which uses, as a framework, some particular algebras of unbounded operators. Here, as an example, we only want to discuss a situation where the sum of this series exists finite.

Suppose that there exists an integer $K$ and a positive real number $L$ such that the following holds:

$$\sup_{\varphi, \Psi \in \mathcal{D}} | < \varphi, [H, x_0]_K \Psi > | = L. \tag{4.2}$$

Then, since $H$ maps $\mathcal{D}$ into itself, calling $\varphi_H = H\varphi$ and $\Psi_H = H\Psi$ we get

$$| < \varphi, [H, x_0]_{K+1}\Psi > | \leq | < \varphi_H, [H, x_0]_K \Psi > | + |$$
$$< \varphi, [H, x_0]_K \Psi_H > | \leq 2L,$$

which implies that, for all positive integers $n$,

$$\sup_{\varphi, \Psi \in \mathcal{D}} | < \varphi, [H, x_0]_{K+n} \Psi > | \leq 2^n L.$$

This estimate ensures the existence of $\sum_{k=0}^{\infty} \frac{(it)^k}{k!} < \varphi, [H, x_1^o]_k \Psi >$, for any $\varphi, \Psi \in \mathcal{D}$. To see this, it is enough to split the sum in two contributions, the first $\sum_{k=0}^{K-1} \frac{(it)^k}{k!} < \varphi_H, [H, x_1^o]_k \Psi >$, which exists since it is a finite sum, and the second $\sum_{k=K}^{\infty} \frac{(it)^k}{k!} < \varphi_H, [H, x_1^o]_k \Psi >$, which converges because of the previous estimate.

Of course, our working assumption (4.2) is very strong, but it has been chosen here since it gives an idea of the kind of problems which usually arise in QM, as well as of the possible ways to overcome these problems.

The relation between the *classical* solution $\vec{x}_{cl}(t)$ of system (4.1) and the operatorial solution $\vec{x}_{op}(t)$ which is produced by our approach is now evident: taken any two functions $\varphi$ and $\Psi$ in $\mathcal{D}$, the following equality holds:

$$< \varphi, \vec{x}_{cl}(t)\Psi > = < \varphi, \vec{x}_{op}(t)\Psi >, \quad \forall t \in \boldsymbol{R}. \tag{4.3}$$

As for the formalization of the second quantized approach, this is a standard topics in QM and we will not discuss it here. We refer to (Bratteli and Robinson, 1979a,b) for all the details.

Let us now consider briefly the role of the symmetries in our approach. As in the standard literature, we say that a map $U$ acting on a solution $(x_1(t), x_2(t))$ of system (4.1) is a *symmetry* if $U(x_1(t), x_2(t))$ is again a solution of the same system. We can prove the following statement:

**Proposition 4.1.** *Let* $(x_1(t), x_2(t)) = e^{iHt}(x_1^o, x_2^o)e^{-iHt}$ *be a solution of system (4.1) corresponding to the initial conditions* $(x_1(0), x_2(0)) = (x_1^o, x_2^o)$. *Let A be an operator with domain $\mathcal{D}$ for which $e^A$ can be defined. Then:*

*(a)* $(a_1(t), a_2(t)) := e^A(x_1(t), x_2(t))e^{-A}$ *is again a solution of system (4.1) corresponding to the initial conditions* $(a_1(0), a_2(0)) = e^A(x_1^o, x_2^o)e^{-A}$;

*(b) if* $(a_1(t), a_2(t))$ *is another solution of system (4.1), then there exists an operator A with domain $\mathcal{D}$ and such that $e^A$ can be defined, for which* $(a_1(t), a_2(t)) := e^A(x_1(t), x_2(t))e^{-A}$.

*Remark 4.1.* Any self-adjoint operator $A$, bounded or not, is such that $e^A$ is a well defined (bounded or not) operator. But $e^A$ is well defined also under other assumptions, for instance if $A$ is bounded or if $A$ is anti-hermitian.

**Proof:** It is trivial to check the statement (a).

To prove (b), let us suppose that $(x_1(t), x_2(t))$ and $(a_1(t), a_2(t))$ are both solutions of system (4.1). Then it is enough to take $A = i(a_1(0) - x_1(0))p_1 + i(a_2(0) - x_2(0))p_2$, which is defined on $\mathcal{D}$ and which can be exponentiated, to pass from a solution to another. □

Of course, what is contained in this paper is only a first step in the analysis of our procedure. We expect to continue this analysis in a close future. In particular, we believe that the second quantization approach deserves a deeper investigation because of its role in the search of EIoM.

Also, the extension of our strategy to partial differential equations and to more general SODE has to be considered. It is particularly interesting, in our opinion, to analyze the Schrödinger and the Klein-Gordon equations of motion, which describe quantum systems for their own: in these cases, what is our $H$? And how this $H$ is related with the *true* quantum hamiltonian?

Last but not least, in view of possible applications to the determination of EIoM, it must be checked in details the existing literature concerning the construction of operators which commute with a given observable. This kind of problems sometimes appears: for instance, if we construct the algebra $\mathcal{A}$ generated by $H$ and by the identity $\mathbb{1}$, the so-called center of $\mathcal{A}$ is important for us since it contains operators which commute with $H$.

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